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EXTERIOR POWERS AND TORSION FREE MODULES OVER DISCRETE VALUATION RINGS

BY

DAVID M. ARNOLD

ABSTRACT. Pure R-submodules of the p-adic completion of a discrete valuation ring R with unique prime ideal (p) (called purely indecomposable R-modules) have been studied in detail. This paper contains an investigation of a new class of torsion free R-modules of finite rank (called totally indecomposable R-modules) properly containing the class of purely indecomposable R-modules of finite rank. Exterior powers are used to construct examples of totally indecomposable modules.

Introduction. A torsion free (t.f.) R-module A of finite rank is *totally indecomposable* if A is reduced, $\operatorname{Hom}(A, R) = 0$ (i.e. A has no free summands) and every pure submodule of A is either free or indecomposable. The *p-rank* of A is the R/pR dimension of A/pA.

Theorem 1.1. *Suppose that A is a reduced t.f. R-module of finite rank with p-rank n and that* $\operatorname{Hom}(A, R) = 0$. *The following statements are equivalent*: (a) *Every t.f. quotient of A is either divisible or indecomposable*; (b) *A is totally indecomposable*; (c) *Every pure submodule B of A with p-rank $B < n$ is a free R-module.*

A t.f. R-module A of finite rank is *co-purely indecomposable* (c.p.i.) if A is reduced, $\operatorname{Hom}(A, R) = 0$, and rank A − p-rank $A = 1$. A consequence of [1] and Theorem 1.1 is that every c.p.i. module is totally indecomposable.

Theorem 1.7. *If A is a totally indecomposable R-module and* rank $A \geq 2(p\text{-rank } A) - 1$, *then the endomorphism ring of A is a local ring.*

Theorem 1.7 has an immediate corollary: If $M = A_1 \oplus \cdots \oplus A_m$, where each A_i is totally indecomposable and rank $(A_i) \geq 2(p\text{-rank } A_i) - 1$ for $1 \leq i \leq n$, then any two direct sum decompositions of M have isomorphic refinements (Azumaya's theorem, e.g., see Lambek [8]).

Theorem 2.1. *Assume that A is a reduced t.f. R-module of finite rank with*

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p-rank n and that $\operatorname{Hom}(A, R) = 0$. *If the nth exterior power of A is reduced, then A is totally indecomposable.*

If A is totally indecomposable and if $0 \neq f \in E(A)$, the endomorphism ring of A, then f is a monomorphism (Corollary 1.3).

Theorem 2.2. *Suppose that A is a reduced t.f. R-module of finite rank with p-rank n, that* $\operatorname{Hom}(A, R) = 0$, *and that every $0 \neq f \in E(A)$ is a monomorphism. If $f, g \in E(A)$ and if $\bigwedge^n f = \bigwedge^n g$, then $f = rg$, where r is an nth root of unity of R.*

Some applications of Theorem 2.2 to totally indecomposable modules are summarized in Corollary 2.3.

Theorem 2.1 and a slightly simplified version of the classical Kurosch matrix invariants are used to construct examples of totally indecomposable modules. Given positive integers n and k, there is a t.f. Z_p-module A with p-rank n, rank $n + k$ such that $\bigwedge^n A$ is reduced (Z_p is the localization of the ring of integers at a prime p). It is true (but we omit the proof) that the set of isomorphism classes of Z_p-modules, with p-rank n, rank $n + k$, such that $\bigwedge^n A$ is reduced, is uncountable.

The converses of Theorems 1.7 and 2.1 are, in general, false (Examples 3.2 and 3.3). Example 3.5 illustrates that the inequality, rank $A \geq 2(p\text{-rank } A) - 1$, of Theorem 1.7 is best possible.

If A and B are c.p.i. modules of p-rank n, then A and B are quasi-isomorphic iff the nth exterior powers of A and B are isomorphic (e.g., see [1]). Example 3.4 illustrates that the analogous statement for totally indecomposable modules is, in general, false.

One can readily prove that a reduced t.f. R-module A of finite rank is totally indecomposable iff every pure submodule of A is either free or *strongly indecomposable* (i.e. has no quasi-direct summands). Consequently, Theorem 1.1 is true if the word "indecomposable" is replaced by "strongly indecomposable". Further, if F is the duality given in [1], then A is totally indecomposable iff FA is totally indecomposable.

0. **Preliminaries.** All modules are assumed to be t.f. R-modules unless otherwise specified. We write $\operatorname{Hom}(A, B)$ for $\operatorname{Hom}_R(A, B)$. Basic references are Kaplansky [7] and Rotman [9].

Every rank-1 torsion free module is isomorphic to either R or K, the quotient field of R. A submodule B of a module A is *pure* if $p^iA \cap B = p^iB$ for all positive integers i and *reduced* if B has no divisible submodules. If x is an element of the module A, the *height of x in $A(h(x))$* is i if $x \in p^iA \setminus p^{i+1}A$ and ∞ otherwise.

A pure submodule B of a module A is a *basic submodule of* A if B is a free R-module and A/B is divisible. Equivalently, B is a free submodule of A and A/B is t.f. and divisible. Every t.f. module has a basic submodule and any two basic submodules of A are isomorphic. If B is a basic submodule of A, then p-rank A = rank B.

Other properties of rank (r) and p-rank (rp) for t.f. modules of finite rank are: $rp(A) = 0$ iff A is divisible; $rp(A) \leq r(A)$; $rp(A) = r(A)$ iff A is free; if $0 \to A \to B \to C \to 0$ is an exact sequence, then $rp(B) = rp(A) + rp(C)$ and $r(B) = r(A) + r(C)$; and if $B \subset A$ and A/B is torsion, then $rp(A) \leq rp(B)$ and $r(A) = r(B)$.

A t.f. R-module A of finite rank with p-rank 1 is a purely indecomposable (p.i.) module iff A is reduced. Furthermore, $E(A)$, the endomorphism ring of A, is a commutative local ring with $pE(A)$ as the ideal of nonunits.

1. Totally indecomposable modules.

Proof of Theorem 1.1. (a) $\Rightarrow$ (b) Assume that B is a nonfree pure submodule of A, that $rp(B) < n$ and that C is a basic submodule of B. Then $rp(A/C) > 0$, since $rp(B) < n = rp(A)$. Note that B/C is divisible, hence a summand of A/C. Therefore, A/C is a nondivisible t.f. quotient of A with a nonzero proper summand, a contradiction to (a).

Consequently, every nonfree pure submodule of A has p-rank n. Let $B = C \oplus D$ be a nonfree pure submodule of A. Then $rp(B) = rp(C) + rp(D) = n$, so $rp(C) \leq n$ and $rp(D) \leq n$. It now follows that either $C = 0$ or $D = 0$ since C and D are pure submodules of A, A is reduced, and B is not a free R-module.

(b) $\Rightarrow$ (c) Suppose that C is a pure submodule of A with $rp(C) < n$ and that B is a basic submodule of C. There is a nonzero element $x + B$ of A/B such that $x + B$ has zero height in A/B, since $rp(C) < n = rp(A)$. Therefore, $R(x + B)$ is a pure submodule of A/B and $B \oplus Rx$ is a pure submodule of A.

We prove that $C \oplus Rx$ is a pure submodule of the totally indecomposable module A, which implies that $C \oplus Rx$, hence C, is a free R-module. Clearly, $C \cap Rx = 0$. Assume that $a \in A$ and that $p^i a = c + rx \in C \oplus Rx$. Since C/B is divisible, there is $c' \in C$, $b \in B$ with $c = p^i c' + b$. Thus, $p^i(a - c') = b + rx \in B \oplus Rx$. Since $B \oplus Rx$ is a pure submodule of A, $a - c' \in B \oplus Rx$ and $a \in C + (B \oplus Rx) \subset C \oplus Rx$.

(c) $\Rightarrow$ (a) Let $f: A \to C$ be an epimorphism, where C is t.f. and not divisible, i.e. $rp(C) > 0$. Then $rp(\ker f) < n$ so by (c), $\ker f$ is free, i.e. $rp(\ker f) = r(\ker f)$. Therefore, $r(C) - rp(C) = r(A) - rp(A) = k$. Suppose that $C = C_1 \oplus C_2$, where $rp(C_1) > 0$. Define $g: A \to C_1$ to be the composite of f and the projection of C onto C_1. By the preceding remarks, $\ker g$ is free and $r(C_1) - rp(C_1) = k$. But $k = r(C) - rp(C) = r(C_1) - rp(C_1) + r(C_2) - rp(C_2)$, so $r(C_2) = rp(C_2)$. If $C_2 \neq 0$, then C_2 is a free R-module contradicting the assumption that $\mathrm{Hom}(A, R) = 0$. Therefore, $C_2 = 0$.

Corollary 1.2. *If A is totally indecomposable, then every nonfree pure submodule and every nondivisible t.f. quotient of A is totally indecomposable.*

Corollary 1.3. *Assume that B is a reduced t.f. R-module with* $\mathrm{Hom}(B, R) = 0$, *and that A is totally indecomposable. If $0 \neq f \in \mathrm{Hom}(B, A)$, then $rp(\ker f) \leq rp(B) - rp(A)$. In particular, $rp(B) = rp(A)$ implies that f is monic.*

Proof. Let C be the pure submodule of A generated by $f(B)$. Since $\mathrm{Hom}(B, R) = 0$, $f(B)$ and C are not free R-modules. By Theorem 1.1, $rp(C) = rp(A)$, and $rp(C) \leq rp(f(B))$ since $C/f(B)$ is torsion. Finally, $rp(\ker f) \leq rp(B) - rp(A)$ since $rp(\ker f) + rp(f(B)) = rp(B)$. The last statement of the corollary follows from the assumption that B is reduced.

Corollary 1.4. *Suppose that A is a reduced t.f. R-module of finite rank with p-rank n and that* $\mathrm{Hom}(A, R) = 0$. *Then A is totally indecomposable iff $Rx_1 \oplus \cdots \oplus Rx_n$ is a basic submodule of A for every R-independent subset $\{x_1, \cdots, x_n\}$ of A with $x_i \in A \setminus pA$ for $1 \leq i \leq n$.*

Proof. ($\Rightarrow$) Let $X = \{x_1, \cdots, x_n\} \subset A/pA$ be an R-independent subset of A. Then B, the pure submodule of A generated by X, is free (Theorem 1.1). Moreover, $B = Rx_1 \oplus \cdots \oplus Rx_n$ since $X \subset A \setminus pA$. Finally, A/B is divisible since $rp(A/B) = rp(A) - rp(B) = 0$.

($\Leftarrow$) Let C be a pure submodule of A with $l = rp(C) < n$. If $r(C) \geq n$, let $Rx_1 \oplus \cdots \oplus Rx_l$ be a basic submodule of C and choose $x_{l+1}, \cdots, x_n \in C \cap (A \setminus pA)$ with $\{x_1, \cdots, x_n\}$ an R-independent subset of A. The hypotheses guarantee that $B = Rx_1 \oplus \cdots \oplus Rx_n$ is a basic submodule of A. Consequently, B is a basic submodule of C, contradicting the assumption that $rp(C) < n$.

Assume that $m = r(C) < n$, and choose $x_{m+1}, \cdots, x_n \in A$ such that C', the pure submodule of A generated by C and $\{x_{m+1}, \cdots, x_n\}$, has rank n. Then $rp(C') = n$, by the preceding remarks, so C' is a free R-module. Therefore, C is free and A is totally indecomposable by Theorem 1.1.

Recall that f is a unit in $E(A)$ iff f is an automorphism of A.

Lemma 1.5. *Let A be a t.f. R-module of finite rank with p-rank n, and let B be a basic submodule of A. Then f is a unit in $E(A)$ iff f is a monomorphism and $f(B)$ is a pure submodule of A.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i} & A & \xrightarrow{\Pi} & A/B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \\ 0 & \longrightarrow & f(B) & \xrightarrow{i} & A & \xrightarrow{\sigma} & A/f(B) & \longrightarrow & 0 \end{array}$$

with exact rows where i is an inclusion map, Π and σ are quotient maps and $f'(a+B) = f(a) + f(B)$. Observe that f' is monic since f is monic. Furthermore, $f(B)$ is a pure free submodule of A of rank $n = p$-rank A so $f(B)$ is a basic submodule of A. Therefore, A/B and $A/f(B)$ are t.f. divisible R-modules of rank $k < \infty$ so f' is epic.

A routine diagram chase proves that f is epic, hence f is a unit in $E(A)$.

Define $A_f = \{x \in A \mid x = 0 \text{ or } x \neq 0 \text{ and } h(f(x)) > h(x)\}$, where $f \in E(A)$ and $h(x)$ is the height of x in A, as defined in §0. It is easy to prove that $A_f = 0$ iff f is a unit in $E(A)$.

Lemma 1.6. *Suppose that A is a totally indecomposable R-module of p-rank $n > 1$, rank $n + k$, and that $f \in E(A)$.*

(a) *A_f is a pure submodule of A;*

(b) *if f is a nonunit of $E(A)$, then* $\operatorname{rank}(A_f) \geq k + 1$.

Proof. (a) The elementary properties of height suffice to prove that $RA_f \subset A_f$ and that if $x \in A$ and if $p^i x \in A_f$, then $x \in A_f$.

Assume that $x, y \in A_f$ and that $x + y \neq 0$. If $Rx \cap Ry \neq 0$, then $rx = sy$ for some nonzero $r, s \in R$. Consequently, $x + y \in A_f$. Therefore, assume that $Rx \cap Ry = 0$. Then B, the pure submodule of A generated by x and y, is free (Theorem 1.1). In particular, $B = Rx' \oplus Ry'$ where $h(x') = h(y') = 0$ and $p^l x' = x$, $p^m y' = y$. It now follows that $h(x + y) = \min(h(x), h(y))$.

Observe that f is monic (Corollary 1.3) so $Rx \cap Ry = 0$ implies that $Rf(x) \cap Rf(y) = 0$. Therefore, $h(f(x) + f(y)) = \min(h(f(x)), h(f(y)))$ by the preceding remarks. Finally, $h(f(x) + f(y)) = \min(h(f(x)), h(f(y))) > \min(h(x), h(y))$ so $x + y \in A_f$.

(b) Let $\{x_1, \cdots, x_{n+k}\}$ be a maximal R-independent subset of A. Write $x_i = p^{l_i} y_i$, where $h(y_i) = 0$ for $1 \leq i \leq n + k$. Then $\{y_1, \cdots, y_{n+k}\}$ is an R-independent subset of A and $y_i \in A \setminus pA$ for $1 \leq i \leq n + k$.

Let $B_1 = Ry_1 \oplus \cdots \oplus Ry_n$, a pure submodule of A by Corollary 1.4. By Lemma 1.5, there is $0 \neq z_1 \in B_1$ such that $h(f(z_1)) > z_1$, i.e. $z_1 \in A_f$. In fact, we may choose $z_1 = y_j$, for some j.

Define $B_{i+1} = (B_i \oplus Ry_{n+i})/Rz_i$ a basic submodule of A. Then there is $z_{i+1} = y_j \in A_f$, for some $y_j \notin Rz_1 \oplus \cdots \oplus Rz_i$, by the preceding remarks. Consequently, $\{z_1, z_2, \cdots, z_{k+1}\}$ is an R-independent subset of A_f.

A ring S is *local* if the nonunits of S form an ideal of S, i.e. S is local iff the sum of two nonunits of S is again a nonunit of S.

Proof of Theorem 1.7. If p-rank $A = 1$, then $E(A)$ is local. So assume that p-rank $A = n > 1$, that f and g are nonunits of $E(A)$ and that $f + g = e$ is a unit of $E(A)$. Then $\operatorname{rank}(A_f) \geq k + 1$ and $\operatorname{rank}(A_g) \geq k + 1$, where p-rank $A = n$, rank $A = n + k$ (Lemma 1.6(b)). Consider the exact sequence

$$0 \longrightarrow A_f \cap A_g \longrightarrow A_f \oplus A_g \xrightarrow{\phi} A_f + A_g \longrightarrow 0$$

where $\phi(x+y) = x - y \in A$. Then $\operatorname{rank}(A_f \cap A_g) \geq 2k + 2 - n - k \geq 1$. Since A_f and A_g are pure submodules of A (Lemma 1.6.a), $A_f \cap A_g$ is a pure submodule of A. Choose $0 \neq x \in A_f \cap A_g$ with $x \in A \backslash pA$. Then $f(x) \in pA$ and $g(x) \in pA$, hence $(f+g)(x) = e(x) \in pA$. Therefore, $x \in A_e = 0$ since e is a unit of $E(A)$. This is a contradiction to the choice of x, thus $E(A)$ is local.

Note that if f and g are nonunits of $E(A)$, then $f + g = e$ is a unit of A iff $A_f \cap A_g = 0$. Example 3.5 demonstrates that this may happen if rank $A <$ $2(p\text{-rank } A) - 1$.

2. **Exterior powers.** For $n \geq 2$, the *nth exterior power* of an R-module A, $\bigwedge^n A$, is given by $\bigwedge^n A = (\bigoplus^n A)/N$ where N is the submodule of $\bigotimes^n A$ generated by $\{a_1 \otimes \cdots \otimes a_n | a_i = a_j \text{ for some } i \neq j\}$. The R-module $\bigwedge^n A$ is generated by all elements of the form $a_1 \wedge \cdots \wedge a_n = a_1 \otimes \cdots \otimes a_n + N$. Define $\bigwedge' A = A$ and $\bigwedge^0 A = R$.

The following facts are used in the sequel: Let A, B, and C be t.f. R-modules:

(1) $$\bigwedge^n (B \oplus C) \simeq \sum_{i=0}^{n} \oplus \left(\bigwedge^i B \otimes \bigwedge^{n-i} C \right) \quad \text{(Bourbaki [3]);}$$

(2) $\bigwedge^n$ is a functor: if $f: A \longrightarrow B$,

$$\bigwedge^n f(a_1 \wedge \cdots \wedge a_n) = f(a_1) \wedge \cdots \wedge f(a_n) \quad \text{(Bourbaki [3]);}$$

(3) If $f: A \longrightarrow B$ is monic, then $\bigwedge^n f: \bigwedge^n A \longrightarrow \bigwedge^n B$ is monic (Flanders [4]);

(4) If B is a pure (basic) submodule of A, then $\bigwedge^n B$ is isomorphic to a pure (basic) submodule of $\bigwedge^n A$. If $r(A) = l$ and $rp(A) = m$, then $r(\bigwedge^n A) = C_{l,n}$ and $rp(\bigwedge^n A) = C_{m,n}$ where $C_{i,j}$ is a binomial coefficient (Arnold [1]).

Proof of Theorem 2.1. Let B be a pure submodule of A with $m = rp(B) < n = rp(A)$ and let $C = B \oplus E \subset A$ where $E = Rx_{m+1} \oplus \cdots \oplus Rx_n$. If C is not free, then $\bigwedge^{m+1} C$ is a nonzero divisible R-module (by (4)). Furthermore, $(\bigwedge^{m+1} C) \otimes (\bigwedge^{n-m-1} E)$ is isomorphic to a submodule of $\bigwedge^n A$ by (1) and (3). Now $r(E) = n - m > 0$, so $\bigwedge^{n-m-1} E$ is a nonzero R-module. Thus $\bigwedge^{m+1} C \otimes \bigwedge^{n-m-1} E$ is a nonzero divisible submodule of $\bigwedge^n A$, contradicting the assumption that $\bigwedge^n A$ is reduced.

Consequently, C and B are free R-modules. Now apply Theorem 1.1.

Theorem 2.2. *Assume that A is a reduced t.f. R-module of p-rank n, rank $n + k$, that* $\operatorname{Hom}(A, R) = 0$ *and that every $0 \neq f \in E(A)$ is a monomorphism. If $f, g \in E(A)$ and if $\bigwedge^n f = \bigwedge^n g$, then $f = rg$, where r is an nth root of unity of R.*

Proof. Let $B = Rx_1 \oplus \cdots \oplus Rx_n$ be a free submodule of A. If $f \neq 0$ and

$g \neq 0$, then $f(x_1) \wedge \cdots \wedge f(x_n) = g(x_1) \wedge \cdots \wedge g(x_n) \neq 0$ since $\bigwedge^n f = \bigwedge^n g$ is a monomorphism (cited fact (3)). For $1 \leq j \leq n$, $f(x_j) \wedge g(x_1) \wedge \cdots \wedge g(x_n) = f(x_j) \wedge f(x_1) \wedge \cdots \wedge f(x_n) = 0 \in \bigwedge^{n+1} A$, so $\{f(x_j), g(x_1), \cdots, g(x_n)\}$ is an R-linearly dependent subset of A (Bourbaki [3]). Consequently, $p^l f(B) \subset g(B)$ for some nonnegative integer l.

We prove that if $y \in A$, then there is a nonnegative integer l_y and $0 \neq r_y \in R$ with $p^{l_y} f(y) = r_y g(y)$. Extend y to an R-linearly independent subset $\{y, y_1, \cdots, y_n\}$ of A, observing that rank $A > n$ since A is not free. Define $C_i = Ry \oplus Ry_1 \oplus \cdots \oplus Ry_{i-1} \oplus Ry_{i+1} \oplus \cdots \oplus Ry_n$ for $1 \leq i \leq n$. As a consequence of the preceding paragraph, there is a nonnegative integer l with $p^l f(C_i) \subset g(C_i)$ for $1 \leq i \leq n$. Furthermore, $Ry = C_1 \cap C_2 \cap \cdots \cap C_n$. Thus, $p^l f(Ry) = p^l f(C_1 \cap C_2 \cap \cdots \cap C_n) \subset g(C_1) \cap g(C_2) \cap \cdots \cap g(C_n) = g(Ry)$ (noting that g is monic) and $p^l f(y) = r_y g(y)$ for some $r_y \in R$.

There is $0 \neq y \in A$ with $f(y) = rg(y)$ where r is some nth root of unity of R. Let $Rx_1 \oplus \cdots \oplus Rx_n$ be a free submodule of A and let $y = x_1 + \cdots + x_n$. There are nonnegative integers $l, l_1, \cdots, l_n$ with $p^l f(y) = r_y g(y)$ and $p^{l_i} f(x_i) = r_i g(x_i)$ for some $r_y, r_1, \cdots, r_n \in R$. Let $m = \max\{l, l_1, \cdots, l_n\}$ so that $p^m y = p^m x_1 + \cdots + p^m x_n$. Now $f(p^m y) = p^{m-l} p^l f(y) = p^{m-l} r_y g(y) = p^{m-l} r_y (g(x_1) + \cdots + g(x_n))$ and $f(p^m x_i) = p^{m-l_i} p^{l_i} f(x_i) = p^{m-l_i} r_i g(x_i)$. Therefore, $f(p^m y) = p^{m-l_1} r_1 g(x_1) + \cdots + p^{m-l_n} r_n g(x_n)$. Since g is monic, $\{g(x_1), \cdots, g(x_n)\}$ is an R-independent subset of A, so $p^{m-l} r_y = p^{m-l_i} r_i$ and $r_i = p^{l_i - l} r_y$ for $1 \leq i \leq n$.

Let $q = l_1 + \cdots + l_n$. Since $\bigwedge^n f = \bigwedge^n g$, $z = p^q f(x_1) \wedge \cdots \wedge f(x_n) = p^q g(x_1) \wedge \cdots \wedge g(x_n)$. Moreover, $z = (p^{l_1} f(x_1)) \wedge \cdots \wedge (p^{l_n} f(x_n)) = (r_1 g(x_1)) \wedge \cdots \wedge (r_n g(x_n)) = (r_1 r_2 \cdots r_n) g(x_1) \wedge \cdots \wedge g(x_n)$. Therefore, $p^q = r_1 r_2 \cdots r_n = p^{l_1 + l_2 + \cdots + l_n - nl}(r_y)^n = p^{q-nl}(r_y)^n$, and so $(p^l)^n = (r_y)^n$. Hence $r_y = p^l r$, where $r^n = 1$. But $p^l f(y) = r_y g(y) = p^l r g(y)$ so $f(y) = rg(y)$.

Finally, $f - rg$ is an endomorphism of A with nonzero kernel, so $f = rg$ by assumption.

Corollary 2.3. *Assume that A is a totally indecomposable R-module of odd p-rank n and that 1 is the only nth root of unity of R.*

(a) *The automorphism group of A is isomorphic to a subgroup of the automorphism group of $\bigwedge^n A$;*

(b) *f is a unit in $E(A)$ iff $\bigwedge^n f$ is a unit in $E(\bigwedge^n A)$;*

(c) *if $\bigwedge^n A$ is reduced, then $E(A)$ is commutative;*

(d) *if $E(\bigwedge^n A) = R$, then $E(A) = R$.*

Proof. (a) Let $\mathrm{Aut}(A)$ denote the automorphism group of A. Define p-det: $\mathrm{Aut}(A) \longrightarrow \mathrm{Aut}(\bigwedge^n A)$ by $p\text{-det}(f) = \bigwedge^n f$. Since $\bigwedge^n$ is a functor, p-det is a well-defined group homomorphism. If $f \in$ kernel p-det, then $\bigwedge^n f = 1 = \bigwedge^n 1$. By Theorem 2.2 and Corollary 1.3, $f = 1$. Thus p-det is monic.

(b) ($\Longrightarrow$) $\bigwedge^n$ is a functor.

($\Longleftarrow$) Assume $A_f \neq 0$. There is $x_1 \in A \setminus pA$ with $x_1 \in A_f$, since A_f is a pure submodule of A. Choose $x_2, \cdots, x_n \in A \setminus pA$ such that $B = Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n$ is a basic submodule of $\bigwedge^n A$. Then $\bigwedge^n B = Rx_1 \wedge \cdots \wedge x_n$ is a basic submodule of $\bigwedge^n A$ by cited fact (4). But $\bigwedge^n f(x_1 \wedge \cdots \wedge x_n) \in p \bigwedge^n A$ since $f(x_1) \in pA$. This contradicts the assumption that $\bigwedge^n f$ is a unit.

(c) Since $\bigwedge^n A$ is reduced, $\bigwedge^n A$ is a p.i. module.

Let $f, g \in E(A)$ and note that

$$\bigwedge^n (fg) = \left(\bigwedge^n f\right)\left(\bigwedge^n g\right) = \left(\bigwedge^n g\right)\left(\bigwedge^n f\right) = \bigwedge^n (gf)$$

since $\bigwedge^n$ is a functor and $E(\bigwedge^n A)$ is commutative. By Theorem 2.2 and the assumtion that 1 is the only nth root of unity, $fg = gf$.

(d) If $\bigwedge^n f = r \in R$, then $\bigwedge^n (f^n) = (\bigwedge^n f)^n = r^n = \bigwedge^n r$. By Theorem 2.2, $f^n = r$. Since $\mathrm{Hom}(\bigwedge^n A, \bigwedge^n A) = R$, $\bigwedge^n A$ is reduced. By (b) f is a nonunit of $E(A)$ iff $\bigwedge^n f$ is a nonunit of $E(\bigwedge^n A)$. Thus f is a nonunit iff $f \in pE(A)$, since $f^n = r \in R$.

By (a), the units of $E(A)$ are isomorphic to a subgroup of the units of $\bigwedge^n A$, hence a subgroup of the units of R. It follows that $E(A) = R$.

3. **Examples.** We assume, without further comment, the notation and results of [1].

If A is a t.f. R-module with p-rank n, rank $n + k$, and matrix representative $\left(\begin{smallmatrix} I & \Gamma \\ 0 & I \end{smallmatrix}\right)$, then $\bigwedge^n A$ has a *standard matrix representative* $\left(\begin{smallmatrix} I & \Delta \\ 0 & 1 \end{smallmatrix}\right)$. The matrix $\left(\begin{smallmatrix} \Delta \\ 1 \end{smallmatrix}\right)$ is an R^*-column matrix with $C_{n+k,n}$ elements, obtained by taking the determinants of the $n \times n$ minors of $\left(\begin{smallmatrix} \Gamma \\ I \end{smallmatrix}\right)$.

Example 3.1. Given positive integers n and k, there is a t.f. R-module A with p-rank n, rank $n + k$ such that $\bigwedge^n A$ is reduced.

Proof. Choose A with matrix representative $\left(\begin{smallmatrix} I & \Gamma \\ 0 & I \end{smallmatrix}\right)$, where Γ is a $k \times n$ R^*-matrix and $\{y_{ij} \in \Gamma \mid 1 \leq i \leq k,\ 1 \leq j \leq n\}$ is an algebraically independent set over K. Let $\left(\begin{smallmatrix} I & \Delta \\ 0 & 1 \end{smallmatrix}\right)$ be the standard matrix representative for $\bigwedge^n A$. Then $\left(\begin{smallmatrix} \Delta \\ 1 \end{smallmatrix}\right)$ has K-independent rows, by the choice of Γ, so $\bigwedge^n A$ is reduced.

Example 3.2. Given positive integers n and k, there is an indecomposable (in fact, strongly indecomposable) R-module A of p-rank n, rank $n + k$ such that A is not totally indecomposable. Furthermore, $E(A) = R$.

Proof. We prove the case $n = k = 2$ and leave the general case to the reader. Let $V^* = K^*x_1 \oplus K^*x_2 \oplus K^*x_3 \oplus K^*x_4$ be a K^*-vector space of dimension 4 and choose A with matrix representative $\left(\begin{smallmatrix} I & \Gamma \\ 0 & I \end{smallmatrix}\right)$, where $\Gamma = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$, $\{a, b, c, 1\} \subset R^*$ and $\{a, b, c\}$ is an algebraically independent set over K. In other words, $A = \alpha^*(V) \cap \overline{A}$, where $\alpha^*(V) = Kx_1 \oplus Kx_2 \oplus K(x_3 + ax_1) \oplus K(x_4 + bx_1 + cx_2)$ and

$\overline{A} = R^*x_1 \oplus R^*x_2 \oplus K^*x_3 \oplus K^*x_4$. Then $B = (Kx_1 \oplus K(x_3 + ax_1)) \cap (R^*x_1 \oplus K^*x_3)$ is a pure submodule of A with p-rank 1, rank 2. Consequently, A is not totally indecomposable by Theorem 1.1.

The standard matrix representative of $\bigwedge^2 A$ is $\binom{I\ \Delta}{0\ 1}$, where Δ is the transpose of $(ac, -b, a, -c, 0)$. Thus, $\bigwedge^2 A = D \oplus K$, where D is reduced. If $A = B \oplus C$, then a comparison of ranks and p-ranks shows that $r(B) = r(C) = 2$ and $rp(B) = rp(C) = 1$. Thus $\bigwedge^2 A \simeq \bigwedge^2 B \oplus (B \otimes C) \simeq \bigwedge^2 C \oplus K \oplus (B \otimes C) \oplus K$, a contradiction to the preceding remarks. Therefore, A is indecomposable.

The proof that A is strongly indecomposable follows from the observation that exterior powers preserve quasi-isomorphism and from the preceding argument.

A persistent reader may prove that if $f \in E(A)$ and if $f^*\colon V^* \quad V^*$ is the unique extension of f, then $f^*(\alpha^*(V)) \subset \alpha^*(V)$, $f^*(K^*x_3 \oplus K^*x_4) \subset K^*x_3 \oplus K^*x_4$ and $\{a, b, c\}$ algebraically independent imply that f^* is multiplication by some $r \in R$.

Example 3.3. There is a totally indecomposable R-module A of p-rank 2, rank 4 such that $\bigwedge^2 A$ is not reduced.

Proof. Let A be a t.f. R-module with matrix representative $\binom{I\ \Gamma}{0\ I}$, where $\Gamma = \binom{a\ b}{b\ d}$ and $\{a, b, d\}$ is a subset of R^* which is algebraically independent over K. Then $\bigwedge^2 A$ has $\binom{I\ \Delta}{0\ 1}$ as a standard matrix representative, where Δ is the transpose of $(ad - b^2, -b, a, -d, b)$. Consequently, $\bigwedge^2 A$ is not reduced, since $\{ad - b^2, -b, a, -d, b\}$ is a K-dependent subset of R^*.

Assume that A is not totally indecomposable. By Theorem 1.1, there is a nonfree pure submodule C of A with $rp(C) < 2$. Since A is reduced of rank 4 and p-rank 2, then $r(C) = 2$ or $r(C) = 3$. Let x and y be two R-independent elements of C and let B be the pure submodule of C generated by x and y. Then B is a pure submodule of A with p-rank 1 and rank 2. Hence if A has no pure submodules with p-rank 1 and rank 2, then A is totally indecomposable.

By the preceding remarks, it suffices to prove that if B is a t.f. R-module of p-rank 1, rank 2 and if $0 \neq f \in \operatorname{Hom}(B, A)$, then f is not monic. Let $\binom{1\ e}{0\ 1}$ be a matrix representative for B. Then $B = \beta^*(U) \cap \overline{B}$, where $\beta^*(U) = Kz_1 \oplus K(z_2 \oplus ez_1)$ and $\overline{B} = R^*z_1 \oplus K^*z_2 \subset U^* = K^*z_1 \oplus K^*z_2$. Moreover, $A = \alpha^*(V) \cap \overline{A}$, where $\alpha^*(V) = Kx_1 \oplus Kx_2 \oplus K(x_3 + ax_1 + bx_2) \oplus K(x_4 + bx_1 + dx_2)$ and $\overline{A} = R^*x_1 \oplus R^*x_2 \oplus K^*x_3 \oplus K^*x_4 \subset V^* = K^*x_1 \oplus K^*x_2 \oplus K^*x_3 \oplus K^*x_4$. The map $f\colon B \to A$ has a unique extension $f^*\colon U^* \to V^*$ with $f^*(\overline{B}) \subset \overline{A}$, $f^*(K^*z_2) \subset K^*x_3 \oplus K^*x_4$ and $f\alpha^*(V) \subset \beta^*(U)$. One can now show that $f(z_2) \in Kx_3 \oplus Kx_4$ and that $e \in R$, since $\{a, b, d\}$ is algebraically independent over K. Thus, $B \simeq R \oplus K$. Since A is reduced, $K \subset \ker f$ and so f is not monic.

Example 3.4. There are t.f. R-modules A and B of p-rank 2, rank 4 such that $\bigwedge^2 A$ is reduced, $\bigwedge^2 A \simeq \bigwedge^2 B$, and the modules A and B are not quasi-isomorphic.

Proof. Choose A and B with matrix representatives $\begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & \Delta \\ 0 & I \end{pmatrix}$, respectively, where $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\Delta = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ and $\{a, b, c, d\}$ is a subset of R^*, algebraically independent over K. Since $\bigwedge^2 A$ and $\bigwedge^2 B$ have identical standard matrix representatives, $\bigwedge^2 A \simeq \bigwedge^2 B$. The fact that A and B are not quasi-isomorphic follows from the fact that $\{a, b, c, d\}$ is an algebraically independent set over K.

Note that if F is the duality given in [1], then $B = FA$.

Example 3.5. There is a totally indecomposable R-module A of p-rank 3, rank 4 such that $E(A)$ is not local.

Proof. Let R be the localization of the ring of integers at a prime $p > 3$, and let $f(X) = X^4 + pX^3 + pX^2 + (1 - p)X + p$. Then $f(-1) \equiv 0 \pmod p$ and $f'(-1) \not\equiv 0 \pmod p$ where $f'(X) = 4X^3 + 3pX^2 + 2pX + (1 - p)$ is the derivative of f. Therefore, $f(X)$ has a root a in R^* (e.g., see Bachman [2]). In fact, $a = -1 + pb$, for some $b \in R^*$, so a is a unit in R^*.

Choose A with matrix representative $\begin{pmatrix} 1 & \Gamma \\ 0 & I \end{pmatrix}$, where $\Gamma = (a, a^2, a^3)$, i.e. $A = \alpha^*(V) \cap \overline{A} \subset V^*$, where $V^* = K^*x_1 \oplus K^*x_2 \oplus K^*x_3 \oplus K^*x_4$, $\alpha^*(V) = Kx_1 \oplus Kx_2 \oplus Kx_3 \oplus Ky$, $y = x_4 + ax_1 + a^2x_2 + a^3x_3$, and $\overline{A} = R^*x_1 \oplus R^*x_2 \oplus R^*x_3 \oplus K^*x_4$.

Define $g\colon V^* \longrightarrow V^*$ by

$$
\begin{aligned}
g(x_1) &= x_1 + px_2 + (-p + p^2)x_3, \\
g(x_2) &= (p + pa)x_1 + (1 + pa^2)x_2 + (-p^2 + p + pa^3)x_3 + px_4, \\
g(x_3) &= ax_1 + (p + a^2)x_2 + (1 - p^2 + a^3)x_3 + x_4, \\
g(x_4) &= (p - a^2p - a^3)x_4.
\end{aligned}
$$

Then

$$g(y) = (-p + pa)x_1 + pa^2x_2 + (-p^2 + pa^3)x_3 + px_4,$$

so $g(\alpha^*(V)) \subset \alpha^*(V)$ and $g(\overline{A}) \subset \overline{A}$. Therefore, g, restricted to A, is an endomorphism of A.

Observe that $g(y) \in pA$ and that $y \in A \setminus pA$ since a is a unit in R^*. Thus g is not a unit of $E(A)$ since $A_g \neq 0$. Furthermore, $(1 - g)(x_1) = -px_2 - (p^2 - p)x_3 \in pA$ and $x_1 \in A \setminus pA$ so $1 - g$ is not a unit of $E(A)$. This proves that $E(A)$ is not a local ring.

Finally, we prove that A is totally indecomposable. Note that

$$f(X) \equiv X^4 + X^3 + X^2 + 1 \equiv (X + 1)(X^3 + X + 1) \pmod 2.$$

If $f(X)$ is reducible in $Z[X]$, then $f(X)$ is the product of a linear factor and a cubic factor since $X^3 + X + 1$ is irreducible modulo 2. Clearly, $f(X)$ has no linear factors in $Z[X]$, so $f(X)$ is irreducible in $Z[X]$, hence $Q[X]$, where Q is the field of rational numbers. Therefore, $f(X)$ is an irreducible polynomial in

$R[X]$ and $\{1, a, a^2, a^3\}$ is an R-independent set. It now follows that $\bigwedge^3 A$ is reduced since the standard matrix representative of $\bigwedge^3 A$ is $\begin{pmatrix} I & \Delta \\ 0 & 1 \end{pmatrix}$ where Δ is the transpose of $(a^3, a, -a^2)$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NEW MEXICO 88001